A Method of "Alternating Corrections" for the Numerical Solution of Two-Point Boundary Value Problems

By David A. Pope

Abstract. In this paper a method of "alternating corrections" is defined and analyzed for the numerical solution of the two-point boundary value problem

$$
\begin{aligned}
y'' &= f(x, y) \\
y(0) &= a \\
y(1) &= b.
\end{aligned}
\tag{0.1}
$$

The case where the first derivative does not enter explicitly into the differential equation is chosen for simplicity of treatment. The alternating corrections method can easily be modified to treat the more general case. The function $f(x, y)$ is assumed to have continuous second derivatives, but the differential equation may, of course, be non-linear.

The method to be described is essentially a relaxation technique suitable for an automatic digital computer. The main feature of the method is that most of the "correcting" is done in the early stages of the computation, using a small number of points; thus a rough approximation to the solution is obtained quickly. This approximation can then be made more accurate in the later stages of the computation, as the number of points is increased.

In Section 1 the method is described. Section 2 gives a rigorous truncation and stability analysis. Section 3 contains the proof of the convergence of the method giving an estimate of the rate of convergence, and in Section 4 some experimental results obtained on a digital computer are examined.

1. Definition of the Method. In the following, we will denote by R a closed and bounded region of the x-y plane, in which we will assume both the solution to (0.1) and the approximations to that solution are known to lie, a priori. (See Collatz [1] p. 188 for a sufficient condition for the existence of the solution to this problem.) The function $f(x, y)$ is assumed to be continuous and to have continuous first and second derivatives in R. The method to be discussed consists of two stages, as follows:

A. *Interpolation by Halves.* Suppose the interval [0, 1] is partitioned into n equal parts by $n + 1$ equally spaced points, and an approximation y_j to the solution of (0.1) is defined at these points. We then refine the partition by subdividing [0, 1] into $2n$ equal parts, by $2n + 1$ equally spaced points x_j, $j = 0, 1, 2, \cdots 2n$. Then the points of the original partition are given by the x_j with even index. We then interpolate y_j for the odd indices by using the explicit formula

$$
y_j = \frac{1}{2}[y_{j+1} + y_{j-1}] - \frac{h^2}{4}[f(x_{j+1}, y_{j+1}) + f(x_{j-1}, y_{j-1})], \qquad j = 1, 3, 5, \cdots 2n - 1.
\tag{1.1}
$$

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We shall abbreviate the right-hand side of (1.1) by the operator $K(y_j)$. Here, and in the following, $h = \Delta x_j = 1/2n$.

B. *Alternating Corrections.* After step A, the values of y_j with odd and even index are corrected alternately by using the same formula (1.1). Thus we have

$$y_j^{s+1} = K(y_j^s) \qquad \text{for } j \text{ odd,} \tag{1.2}$$

$$y_j^{s+1} = K(y_j^{s+1}) \qquad \text{for } j \text{ even, } j \neq 0, j \neq 2n, \tag{1.3}$$

$$y_0^{s+1} = a, \tag{1.4}$$

$$y_{2n}^{s+1} = b, \tag{1.5}$$

where y_j^s is the value of y_j at the s-th iteration of step B. Then, as we shall show in Section 3, as $s \to \infty$, the values y_j^s approach the solution of the system of difference equations

$$\begin{aligned} y_j &= K(y_j) \qquad j = 1, 2, \cdots, 2n-1, \\ y_0 &= a \\ y_{2n} &= b. \end{aligned} \tag{1.6}$$

To start the computation, we usually will set $n = 1$, and $y_0 = a$, $y_1 = b$. Then step A interpolates a value at $x = \frac{1}{2}$, and we renumber the values y_0, y_1, y_2. Here step B is not needed, so we perform step A again, getting now five values. At this point we perform step B a number of times, until sufficient convergence to (1.6) for our purpose is obtained. We continue in this way, doubling the number of points with step A, then following this with a number of iterations of step B, until we have the desired accuracy. In Section 3 we will consider some estimate of the number of iterations of step B necessary for a given accuracy.

2. Stability and Truncation Error. In this section we shall give a rigorous estimate of the truncation error in and stability of the difference equations (1.6). We note that the global truncation error is of order h^{-2} times the local truncation error, rather than h^{-1} times, as might be expected from a naive analysis.

In the following, we let $Y(x)$ be the exact solution to the differential equation (0.1). Let $Y_j = Y(x_j)$, and let y_j be the exact solution to the system of difference equations (1.6). Let the error $e_j = Y_j - y_j$. Then, using the law of the mean, we obtain from (1.6) the system of difference equations

$$\begin{aligned} e_j = \frac{1}{2}[e_{j+1} + e_{j-1}] - \frac{h^2}{4}[e_{j+1} fy(x_{j+1}, \eta_{j+1}) \\ + e_{j-1} fy(x_{j-1}, \eta_{j-1})] + t_j, \qquad j = 1, 2, \cdots 2n-1, \end{aligned} \tag{2.1}$$

and

$$e_0 = e_{2n} = 0, \tag{2.2}$$

where η_j is between Y_j and y_j. Here the local truncation error

$$t_j = -\frac{5h^4}{24} y^{iv}(\xi_j), \qquad x_{j-1} < \xi_j < x_{j+1},$$

which is obtained from Taylor's formula. Rewriting (2.1) in matrix form, we have

$$Ae = -\frac{h^2}{4} Fe + t \tag{2.3}$$

where e is the column error vector whose transpose is $(e_1, e_2, \cdots e_{2n-1})$, t the truncation error vector with transpose $(t_1, t_2, \cdots t_{2n-1})$, A is the second difference matrix

$$A = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 & \cdots \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \cdots \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots \\ \cdots & & & & \end{bmatrix}$$

and F is the matrix

$$F = \begin{bmatrix} 0 & g_{12} & 0 & 0 & \cdots \\ g_{21} & 0 & g_{23} & 0 & \cdots \\ 0 & g_{32} & 0 & g_{33} & \cdots \\ \cdots & & & & \end{bmatrix}$$

where $g_{ij} = f_y(x_i, \eta_j)$, evaluated at the intermediate points given in (2.1). It is well known that the matrix A has eigenvectors v_j with components $\sin \frac{jm\pi}{2n}$, $m = 1, 2, \cdots 2n - 1$, and corresponding eigenvalues $\lambda_j = 1 - \cos \frac{j\pi}{2n}$ $j = 1, 2; \cdots 2n - 1$. Hence A^{-1} exists, and its largest eigenvalue is $\left(1 - \cos \frac{\pi}{2n}\right)^{-1}$. Therefore, multiplying (2.3) by A^{-1}, we have

$$\left(I + \frac{h^2}{4} A^{-1}F\right) e = A^{-1}t. \tag{2.4}$$

Now we can prove two lemmas giving estimates of the error.

LEMMA 1. *Suppose $f_y > 0$ in the region R. Then the components of the error vector e satisfy the inequality*

$$| e_i | \leqq \frac{5h^2 \max | y^{iv} |}{12 \min f_y} \qquad i = 1, 2, \cdots 2n - 1, \tag{2.5}$$

where the extreme values are taken over the region R.

Proof. Let the norm $\| e \| = \max_i | e_i |$, and the subordinate matrix norm $\| A \| = \max_i \sum_j | a_{ij} |$. (Cf. Faddeeva [2] p. 58.) Rewriting (2.3) in the form

$$\left(A + \frac{h^2}{4} F\right) e = (I - B)\, e = t, \tag{2.6}$$

we have defined the matrix $B = I - A - \frac{h^2}{4} F$. But since $f_y > 0$, we can conclude that $\| B \| \leqq 1 - \frac{h^2}{2} \operatorname{Min} f_y < 1$, where the minimum is taken over the closed region R. Hence the series $I + B + B^2 + \cdots$ converges to $(I - B)^{-1}$, and $\| (I - B)^{-1} \| \leqq (1 - \| B \|)^{-1}$. Also we note that $\| t \| \leqq \frac{5h^4}{24} \max | y^{iv} |$.

Using these estimates and (2.6) gives us

$$\| e \| \leqq \frac{\| t \|}{1 - \|B\|} \leqq \frac{5h^4}{24} \max | y^{iv} | \cdot \frac{2}{h^2 \min f_y}$$

from which inequality (2.5) follows.

The second lemma takes care of the case when f_y is negative or zero in R. Here we estimate the root-mean- square of the error.

Lemma 2. *Suppose $| f_y | < \pi^2$ in the region R. Then, for h sufficiently small, the error vector satisfies*

$$\frac{\| e \|}{\sqrt{2n - 1}} \leqq \frac{5h^2 \max | y^{iv} |}{12\pi^2(1 - \pi^{-2}\max | f_y |)} + O(h^4). \tag{2.7}$$

Proof. In this proof, we use the euclidean norm, and the subordinate matrix norm $\|A\|$ equal to the square root of the largest eigenvalue of AA^T (Faddeeva [2] p. 59). We see by inspection that FF^T has a maximum row sum not exceeding $\max 4f_y^2$. But we know that its largest eigenvalue does not exceed this maximum row sum. This gives the estimate

$$\|F\| < 2 \max | f_y | . \tag{2.8}$$

Also, from the eigenvalues of the symmetric matrix A we obtain

$$\|A^{-1}\| = (1 - \cos \pi h)^{-1}. \tag{2.9}$$

Using this together with (2.8) we get

$$\left\| \frac{h^2}{4} A^{-1}F \right\| \leqq \frac{h^2 \max | f_y |}{2(1 - \cos \pi h)} . \tag{2.10}$$

But for small h, we have the estimate

$$(1 - \cos \pi h)^{-1} = \frac{2}{\pi^2 h^2} - \frac{1}{6} + O(h^2), \tag{2.11}$$

hence, for h sufficiently small,

$$\left\| \frac{h^2}{4} A^{-1}F \right\| \leqq (\pi^{-2} + O(h^2)) \max | f_y | < 1, \tag{2.12}$$

since $\max | f_y | < \pi^2$. Also, with this norm, $\|t\| \leqq \frac{5}{24} \operatorname{Max} | y^{iv} | \sqrt{2n + 1}$.

Therefore, by the same reasoning as in Lemma 1, $\left(I + \frac{h^2}{4} A^{-1}F\right)^{-1}$ exists, and its norm does not exceed $\left(1 - \left\| \frac{h^2}{4} A^{-1}F\right\|\right)^{-1}$. Putting these estimates into (2.4), we get (2.7), proving the lemma.

It should be noted that the restriction $-f_y < \pi^2$ is a natural one, as can be seen from an examination of the boundary value problem

$$(2.13)\qquad \begin{aligned} y'' &= -Ky \\ y(0) &= 0 \\ y(1) &= 1. \end{aligned}$$

Here, of course, the solution does not exist when $K = -f_y = \pi^2$, as we have an eigenvalue problem.

3. Convergence of the Alternating Corrections Method. In this section we shall give a proof of the convergence of the alternating corrections method (method B of Section 1). Following the notation of Section 1, we shall denote the value of the approximation at the point x_j for the s-th iteration by y_j^s, and the exact solution to the difference equations (1.6) by y_j. Then we define the error $\epsilon_j^s = y_j - y_j^s$. Then we have, for j odd,

$$(3.1)\qquad \epsilon_j^{s+1} = \frac{1}{2}[\epsilon_{j+1}^s + \epsilon_{j-1}^s] - \frac{h^2}{4}[\epsilon_{j+1}^s f_y(x_{j+1}, \eta_{j+1}) + \epsilon_{j-1}^s f_y(x_{j-1}, \eta_{j-1})],$$

where η_j is between y_j and y_j^s, for each j. For j even, $j \neq 0$, $j \neq 2n$,

$$(3.2)\qquad \epsilon_j^{s+1} = \frac{1}{2}[\epsilon_{j+1}^{s+1} + \epsilon_{j-1}^{s+1}] - \frac{h^2}{4}[\epsilon_{j+1}^{s+1} f_y(x_{j+1}, \xi_{j+1}) + \epsilon_{j-1}^{s+1} f_y(x_{j-1}, \xi_{j-1})],$$

and for the endpoints,

$$(3.3)\qquad \epsilon_0^s = \epsilon_{2n}^s = 0.$$

Now we define $\mu = \max\left|1 - \frac{h^2}{2} f_y\right|$, the maximum being taken over the region R. Then for j odd, we have

$$(3.4)\qquad |\epsilon_j^{s+1}| \leqq \tfrac{1}{2}\mu[|\epsilon_{j+1}^s| + |\epsilon_{j-1}^s|]$$

and for j even, $j \neq 0$, $j \neq 2n$

$$(3.5)\qquad |\epsilon_j^{s+1}| \leqq \tfrac{1}{2}\mu[|\epsilon_{j+1}^{s+1}| + |\epsilon_{j-1}^{s+1}|].$$

To estimate the error ϵ_j^s, we majorize it with a quantity E_j^s, defined recursively as follows:

$$(3.6)\qquad E_j^0 = |\epsilon_{2j}^0| \qquad j = 0, 1, \cdots n.$$

$$(3.7)\qquad E_j^{s+1} = \tfrac{1}{4}\mu^2[E_{j-1}^s + 2E_j^s + E_{j+1}^s] \qquad j = 1, 2, \cdots n-1.$$

$$(3.8)\qquad E_0^{s+1} = E_n^{s+1} = 0.$$

In this section we will use the euclidean norms $\|\epsilon\|^2 = \sum_0^{2n} \epsilon_j^2$ and

$$\|E\|^2 = \sum_0^m E_j^2.$$

Lemma 3 gives an estimate for $\|\epsilon\|$ in terms of $\|E\|$.

LEMMA 3. *For every* $s \geqq 1$,

$$\| \epsilon^s \|^2 \leqq (1 + \mu^2) \| E^s \|^2. \tag{3.9}$$

Proof. First the inequality $| \epsilon_{2j}^s | \leqq E_j^s$ is established by induction on s. For $s = 0$, the inequality holds by definition. Assuming the inequality is true for s, the proof for $s + 1$ follows from the inequalities

$$\begin{aligned} | \epsilon_{2j}^{s+1} | &\leqq \tfrac{1}{2}\mu[| \epsilon_{2j-1}^{s+1} | + | \epsilon_{2j+1}^{s+1} |] \\ &\leqq \tfrac{1}{4}\mu^2[| \epsilon_{2j-2}^{s} | + 2 | \epsilon_{2j}^{s} | + | \epsilon_{2j+2}^{s} |] \\ &\leqq E_j^{s+1}. \end{aligned}$$

Now for $2j + 1$ we have the inequality

$$\begin{aligned} | \epsilon_{2j+1}^{s} | &\leqq \tfrac{1}{2}\mu[| \epsilon_{2j}^{s} | + | \epsilon_{2j+2}^{s} |] \\ &\leqq \tfrac{1}{2}\mu[E_j^s + E_{j+1}^s]. \end{aligned}$$

Combining these, and using the triangle inequality, we obtain

$$\begin{aligned} \| \epsilon^s \|^2 &\leqq \sum_0^n | E_j^s |^2 + \frac{1}{4}\mu^2 \sum_0^{n-1} | E_j^s + E_{j+1}^s |^2 \\ &\leqq (1 + \mu^2) \|E^s \|^2, \end{aligned} \tag{3.10}$$

which proves Lemma 3.

We now expand E_j^s in a finite Fourier sine series, with Fourier coefficients given by

$$F_m^s = \frac{2}{n} \sum_{j=1}^{n-1} E_j^s \sin \frac{mj\pi}{n}, \qquad m = 1, 2, \cdots n - 1. \tag{3.11}$$

Then we have the expansion

$$E_j^s = \sum_{m=1}^{n-1} F_m^s \sin \frac{mj\pi}{n}, \qquad j = 0, 1, \cdots, n. \tag{3.12}$$

Substitution of (3.12) into (3.7) now yields the recursion relation

$$F_m^{s+1} = \frac{1}{2}\mu^2 \left(1 + \cos \frac{m\pi}{n}\right) F_m^s. \tag{3.13}$$

From (3.13) we then obtain the estimate

$$\| E^s \| \leqq \left[\frac{1}{2}\mu^2 \left(1 + \cos \frac{\pi}{n}\right)\right]^s \| E^0 \|, \tag{3.14}$$

and applying Lemma 3, we have the final estimate

$$\| \epsilon^s \| \leqq (1 + \mu^2)^{1/2} \left[\frac{1}{2}\mu^2 \left(1 + \cos \frac{\pi}{n}\right)\right]^s \| \epsilon^0 \|. \tag{3.15}$$

Therefore we have proved

LEMMA 4. *If* $-1 < \rho = \frac{1}{2}\mu^2\left(1 + \cos \frac{\pi}{n}\right) < 1$, *the alternating corrections method* *will converge geometrically to the solution of the difference equations* (1.6), *with convergence factor* ρ.

For small h, we note that

$$\rho = \max\,[1 - h^2(f_y + \pi^2) + O(h^4)] \tag{3.16}$$

and again we have the natural restriction for convergence mentioned at the end of Section 2.

The number of iterations needed at each stage in the method can now be estimated as follows. Clearly, the convergence factor increases as $h \to 0$, hence the convergence is much faster for large values of h. On the other hand, it would be futile to carry the iterations so far that $\| \epsilon \|$ is much smaller than $\| e \|$, as we are interested not in the solution of (1.6), but of (0.1). Hence a useful compromise might be to iterate the alternating corrections until $\| \epsilon \|$ and $\| e \|$ are approximately equal, then to interpolate, and start again with interval $h/2$. If this scheme is followed, we would want to cut the error ϵ by a factor of about $\frac{1}{4}$ by iteration after each interpolation, since the error $\| e \|$ is of order h^2. Then we have $\rho^s = \frac{1}{4}$, which gives us the approximation

$$s = -\frac{\log 4}{\log \rho} \approx \frac{\log 4}{h^2(f_y + \pi^2)}\,.$$

This shows that it would take about four times as many iterations for the next stage, after interpolation by halves. Since there are about twice as many points, the total amount of computational work is multiplied by eight at each succeeding stage. Clearly this process cannot be used for very many stages.

In practice h will probably not be made less than 2^{-8} or 2^{-9}, and if more accuracy is needed, a more sophisticated set of difference equations than (1.6) would be used. The alternating corrections method, however, is excellent for obtaining

TABLE 1

h	error at $x = \frac{1}{2}$	number of iterations
2^{-1}	.052083	0
2^{-2}	.013021	27
2^{-3}	.003255	100
2^{-4}	.000814	329
2^{-5}	.000203	1026
2^{-6}	.000051	2948

TABLE 2

h	error at $x = \frac{1}{2}$	number of iterations
2^{-1}	.065074	0
2^{-2}	.013935	23
2^{-3}	.003366	82
2^{-4}	.000834	269
2^{-5}	.000208	831
2^{-6}	.000052	2339

a good approximation quickly, which could be used as a first guess in a more complicated relaxation scheme.

4. Some Experimental Results. In this section we shall discuss the results of two problems which were computed using the alternating corrections method. The computation was done using the Univac Scientific 1103 computer at the University of Minnesota Scientific Computing Laboratory.

The first problem was the linear equation

$$\begin{aligned} y'' &= 2x^2 \\ y(0) &= 0 \\ y(1) &= 1. \end{aligned} \tag{4.1}$$

In Table 1 the results of this computation are summarized.

Formula (2.7) with $f_y = 0$, $y^{iv} = 4$ gives the r.m.s. error $< .16887h^2$, in good agreement with the error at $x = \frac{1}{2}$.

For each value of h, stage B was iterated until there was no change larger than 2^{-29} in any y_j^s. The number of iterations necessary to accomplish this is also given in Table 1. If we use the estimate $\| \epsilon^0 \| = .16887h^2$, we get the relation

$$.16887h^2\rho^s = 2^{-29},$$

from which we get the approximation

$$s \approx h^{-2}(1.856 + .2026 \log h),$$

which agrees well with the number of iterations actually performed.

The second problem tried was the nonlinear equation

$$\begin{aligned} y'' &= 2y^2 \\ y(0) &= 0 \\ y(1) &= 1 \end{aligned} \tag{4.2}$$

Table 2 gives a summary of this computation.

The fact that the global truncation error is of order h^2 is again displayed in Table 2. The number of iterations necessary at each stage was governed by the same scheme as in problem 1, but with the criterion 2^{-31} instead of 2^{-29}.

Finally, it may be noted that the approximations generated by the alternating corrections method could be improved greatly by a "deferred approach to the limit", using the approximations obtained from the last two values of h computed.

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University of Minnesota
Minneapolis, Minnesota

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